BPS condensates, matrix models and emergent string theory

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ABSTRACT: A prescription is given for computing anomalous dimensions of single trace operators in SYM at strong coupling and large N using a reduced model of matrix quantum mechanics. The method involves treating some parts of the operators as "BPS condensates" which, in certain limit, have a dual description as null geodesics on the S^5. In the gauge theory, the condensate is similar to a representative of the chiral ring and it is described by a background of commuting matrices. Excitations around these condensates correspond to excitations around this background and take the form of "string bits" which are dual to the "giant magnons" of Hofman and Maldacena. In fact, the matrix model approach gives a *quantum* description of these string configurations and explains why the infinite momentum limit suppresses the quantum effects. This method allows, not only to derive part of the classical sigma model Hamiltonian of the dual string (in the infinite momentum limit), but also its quantum canonical structure. Therefore, it provides an alternative method of testing the AdS/CFT correspondence without the need of integrability.

KEYWORDS: Matrix Models, AdS-CFT Correspondence, Long strings.

Contents

1. Introduction

Understanding the strong coupling limit of non-abelian gauge theories is still an outstanding open problem in theoretical physics. Most of our understanding comes from Conformal Field Theories (CFTs). According to the AdS/CFT conjecture, at large N and large but fixed 't Hooft coupling, we should find an effective geometrical description of these theories in terms of perturbative string theory on an asymptotically AdS background [1]. Proving (or disproving) this conjecture is still an important open problem. However, much evidence in its favor have been found in recent years. The best studied example of the correspondence is the duality between $\mathcal{N} = 4$ SYM theory in four dimensions and string theory on asymptotically $AdS_5 \times S^5$. More precisely, the Hamiltonian and Hilbert space of $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$ are identified with the Hamiltonian and Hilbert space

of the dual "quantum gravity" theory.[1] One of the most significant obstruction to testing the correspondence is our inability to make strong coupling calculations in SYM theory.

In practice, one can calculate the Hamiltonian of SYM theory only to a finite loop order. Using the operator/state correspondence one maps this Hamiltonian to the Dilatation operator of SYM on $\mathbb{R}^{1,3}$. One finds that the Hamiltonian can be written as a model of matrix quantum mechanics. In fact, the complete one-loop model was found in [3]. At higher loops, the form of the Hamiltonian seems to be extremely complicated.

However, even with this limited knowledge, one can perform quite impressive tests of the AdS/CFT correspondence. This is usually done by studying states of the gauge theory with large R-charges. These states are dual to fast rotating strings in the dual gravity theory. The use of these limits was first employed in [4]. These fast string limits can now be used to even match the sigma model action of the dual string (in the appropriate limit). There is an extensive literature on this subject so here we will refer the reader to the recent review [5].

What about higher loops? It turns out that, in the large N limit and when acting on single trace states, the SYM theory Hamiltonian exhibits a remarkable property: *integrability*. This was first observed at one loop by Minahan and Zarembo [6]. Integrability is also observed at two loops and it is argued to persist to all loops. An all-loop guess for the Bethe ansatz has been presented in [7, 8]. In this language, one does not know the Hamiltonian at all loops. Instead, one think of each field in the single trace operator as a "particle" which experiences "scattering" with the other fields of the operator. The S-matrix describing this scattering is what enters the Bethe Ansatz.

Integrability has also been found in the classical string theory [9]. This finding has also been accompanied by a similar guess for the presumed quantum Bethe ansatz for the closed strings [10].[2] One can see that the Bethe equations are very similar on both sides of the correspondence but they do not quite agree. In fact, all the disagreement can be encoded in a single overall phase in the Bethe ansatz' S-matrix [10]. Solving this discrepancy is still an open problem.

In any case, one would like to know the explicit form of the Hamiltonian of SYM theory at strong coupling. Of course, a systematic perturbative resummation is not feasible. Therefore, one looks for a reduced model that can capture the main dynamics of this more complicated system. They key to achieve this goal is to look at states that are nearly supersymmetric.

This idea was originally put forward in [12, 13]. It was argued that the effective dynamics of 1/2 BPS states can be described by a reduced quantum mechanical model of a single matrix in an harmonic oscillator potential which, after diagonalization, can be written as N fermions in an harmonic trap. One can describe semiclassical states of the theory in terms of droplets in the single particle phase space. Reduced quantum mechanical sectors are also known to arise in thermal $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ [14, 15].

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[1]The Hamiltonian of the String Theory is naturally defined as the generator of time translations along the global time defined in the asymptotic AdS_5.

[2]For a recent review about integrability and semiclassical strings see [11].

For the 1/2 BPS states, the reduced model was amazingly confirmed by a SUGRA calculation in [16], where it was found that all the 1/2 BPS solutions in IIB SUGRA can also be classified in terms of droplets in a plane.

Generalizations of the reduced matrix model for 1/4 and 1/8 BPS states were proposed in [17] in terms of multiple matrix models of commuting matrices. The gravity side of the story for these states is still incomplete, but some recent progress has been made in [18]. Nevertheless, some important consistency checks for the proposal in [17] have been put forward recently [19, 20].

One can also construct a reduced matrix model for near-BPS states [21]. One can reproduce the energy of the so-called BMN states to all orders in the t'-Hooft coupling. One also gets a nice geometrical picture of near-BPS states that has been amazingly confirmed by Hofman and Maldacena using a purely classical string theory analysis [22]. We will review this result below.

Therefore, even though we do not have a *proof* that the effective dynamics of scalar operators is given by a reduced matrix model, we have non-trivial evidence that this is indeed the case.

In this article we attempt to clarify the meaning of the matrix model calculations. In particular, we give a precise proposal that relates the matrix model computations to the more familiar operator mixing problem, which is the more familiar language used in the context of integrability. This is done in terms of what we call "BPS condensates" which are summarized in section 2. Instead of doing just doing a saddle point calculation as in [21], in section 3 we we give a quite explicit form of the Hamiltonian of the reduced matrix model. Moreover, our Hamiltonian can be easily adapted for generic 1/2 BPS backgrounds.

We also clarify how to obtain the giant magnons of [22] directly from the matrix model calculation and how one can generalize these to the SU(3) sector of the theory. For the SU(3) sector, which is discussed in section 4, the matching with the dual string theory is more restricted since it turns out that one needs to understand the backreaction to the 1/4 and 1/8 BPS condensates. In any case, we obtain a quantum description of the giant magnons. More importantly, we show directly from the matrix model why the infinite momentum limit of Hofman and Maldacena is really a classical limit. The possible interpretation of magnon bound states in our formalism is briefly discussed in section 3.

We also show how the matrix model calculation gives not only the correct Hamiltonian for the string states (in the infinite momentum limit) but also the canonical structure expected from the string theory dual. This is very encouraging since it would be very desirable to match directly the sigma model of the string theory and its canonical structure rather than having to solve for its spectrum. This would allow generalizations to other less symmetric field theories. In section 5 we discuss the backreaction to the BPS condensates and we explain why our method works in the strong coupling limit. Finally, in the conclusion, we discuss the prospects to relate this procedure to the more familiar Bethe Ansatz.

2. The general idea of BPS condensates

Here we want to summarize the general idea of BPS condensates using the familiar single trace scalar operators of SYM theory. In the next sections we will develop the details of this method and its interpretation in terms of the dual string theory.

When doing perturbative calculations in either the gauge theory or the string dual one always has to choose a classical background configuration to expand upon. In the gauge theory side of the correspondence one can rephrase this as defining the expansion around certain set of operators. To have some kind of "perturbative" control, one wants to expand near protected BPS operators.

Here we want to rephrase this expansion in terms of the effective matrix model mentioned above. Lets start by considering a generic SU(2) single trace operator. One can write this kind of state using the bosonized language introduced in [23, 24]:

$$\text{Tr}(Y Z^{n_1} Y Z^{n_2} Y \cdots) . \tag{2.1}$$

In the next section we argue that these states are described by a quantum mechanical matrix model of two complex matrices. For large n_i one can see the Ys as impurities in an otherwise 1/2 BPS operator $\text{Tr}(Z^n)$. In the excited state (2.1), the Z^{n_i}s look localy like BPS states and we call them *1/2 BPS condensates*. In the matrix model it then makes sense to expand around a background of normal matrices: $[Z, \bar{Z}] = 0$ and $Y = 0$. This "classical" BPS background can be expressed as a distribution of eigenvalues of the matrix Z. For the ground state, we will see that this distribution is simply a circular droplet. The fluctuations δY are called "string bits" and they are dual to commutators between the Y and the BPS condensates in the operator language:

$$\text{Tr}(Z^n) \rightarrow \text{Tr}([Y, Z^{n_1}][Y, Z^{n_2}] \cdots), \quad \sum_i n_i = n . \tag{2.2}$$

The states (2.1) serve as a basis for these excitations. The fluctuation δZ are the back-reaction of the condensate and in this case we will see that they can be integrated out to give an effective action for the transverse excitations on the classical BPS background.

In the dual string theory, we will see that for $n_i \rightarrow \infty$ the 1/2 BPS condensates become classical and localize the ends of the string bits on the boundary of the circular droplet. In the dual String Theory, the boundary of the droplet corresponds to a null geodesic on $\mathbb{R} \times S^1$. This is precisely the interpretation advocated recently in [22]. This will be discussed in detail in the next section.

If the number of Y fields is comparable to the number of Zs, it makes sense to expand around a state of the form:

$$\text{Tr}(\{Z^n Y^m\}) . \tag{2.3}$$

Here the curly brackets denote symmetrization between the letters Y and Z. This state is just a rotation of the 1/2 BPS state. For multi-trace operators, symmetrized states similar to this one are 1/4 BPS [25, 26]. Small excitations around this state will be described by turning on commutators that break the symmetrization,

$$\text{Tr}(\{Z^n Y^m\}) \rightarrow \text{Tr}([Y, \{Z^{n_1} Y^{m_1}\}][Z, \{Z^{n_2} Y^{m_2}\}] \cdots), \text{ etc.} \tag{2.4}$$

One can find a basis for these excitations in analogy to (2.1),

$$\text{Tr}(Y\{Z^{n_1}Y^{m_1}\}Z\{Z^{n_2}Y^{m_2}\}\cdots) \,. \tag{2.5}$$

Since the words $\{Z^{n_i}Y^{m_i}\}$ look like 1/4 BPS states we call them *1/4 BPS condensates*. As in the case of the 1/2 BPS condensates this description is more useful when $n_i, m_i \to \infty$. As we will see, the dual interpretation is that, in this limit, the 1/4 BPS condensates localize the ends of the string bits on null geodesic on $\mathbb{R} \times S^3$.

In the matrix model language we should then expand around a "classical" configuration of commuting normal matrices $[Y, Z] = 0 = [Z, \bar{Z}] = [Y, \bar{Y}] = 0$. This configuration can be one of the 1/4 "droplets" of [17]. For the ground state we get a S^3 distribution of eigenvalues. The fluctuations around this background will correspond to the commutators:

$$\delta Z \simeq [Z, \{Z^{n_i}Y^{m_i}\}] \,, \quad \delta Y \simeq [Y, \{Z^{n_i}Y^{m_i}\}] \,. \tag{2.6}$$

We discuss these fluctuations in section 4.

The generalization to SU(3) states should be obvious by now. For example, in the case where we have many Y and Z fields and a few Xs, we can consider the following basis for the transverse excitations

$$\text{Tr}(X\{Z^{n_1}Y^{m_1}\}X\{Z^{n_2}Y^{m_2}\}\cdots) \,. \tag{2.7}$$

In the matrix model we expand around $[Y, Z] = [Y, \bar{Y}] = [Z, \bar{Z}] = 0$, $X = 0$. In this case a fluctuation δY or δZ amounts to a breaking one the condensates and thus leaving this restricted basis, e.g.

$$\{Z^{n_i}Y^{m_i}\} \to [Z, \{Z^{n_i-1}Y^{m_i}\}] \,. \tag{2.8}$$

On the other hand, a δX fluctuation is just a commutator between an X and one of the condensates.

We can now try to consider states with many Zs (for example) and similar quantities of Y and X. In this case it make sense to expand in a basis like

$$\text{Tr}(XZ^{n_1}YZ^{n_2}\cdots) \,. \tag{2.9}$$

In the matrix model we diagonalize Z and expand around $[Z, \bar{Z}] = 0$, $X = Y = 0$ just like in the SU(2) case.

Finally we can have 1/8 BPS condensates by considering symmetrized combinations $\{X^{n_i}Y^{m_i}Z^{p_i}\}$. In this case we expand around configurations with three normal commuting matrices. The fluctuations are dual to the commutators between any of the fields and the condensates just like for the SU(2) states.

3. 1/2 BPS condensates: the SU(2) sector

In this section we briefly review the 1/2 BPS states and their effective dynamics in terms of a normal matrix model. We then consider generic SU(2) states an set up an effective description in terms of a two matrix model similar to the one in [21]. We then construct the

Hilbert space in terms of the 1/2 BPS condensates. We obtain the canonical commutator relations for these states and explain how to obtain their classical limit. In doing so, we recover the picture of the "string bits" of [17, 24] and explain its precise relation with the "giant magnons" of [22]. Moreover, we match the canonical structure expected from the dual string theory and also its sigma model in the limit of large λ and infinite angular momentum. Finally, we comment on the a possible interpretation of the bound states of [22, 27–29] in terms of the matrix model.

3.1 Review of 1/2 BPS states dynamics

The 1/2 BPS states of SYM theory are described by multitrace operator build out of a single complex scalar: $\mathcal{O}_{n_1 n_2 \cdots} = \text{Tr}(Z^{n_1}) \text{Tr}(Z^{n_2}) \cdots$ (see [30] and references therein). They are eigenstates of the dilatation operator with dimension[3]

$$\hat{\Gamma}|\mathcal{O}_{n_1 n_2 \cdots}\rangle = (n_1 + n_2 + \cdots)|\mathcal{O}_{n_1 n_2 \cdots}\rangle = \hat{J}_z|\mathcal{O}_{n_1 n_2 \cdots}\rangle \; . \qquad (3.1)$$

It is well known that at one loop, the contribution from the D-term in the scalar potential $V_D \sim \text{Tr}|[Z, \bar{Z}]|^2$ to the dilatation operator is canceled by fermion and gauge loops [31]. One can translate this to the dual Hilbert space as,

$$\langle \mathcal{O}_{n_1 n_2 \cdots}| \; \text{Tr}|[Z, \bar{Z}]|^2 |\mathcal{O}_{n_1' n_2' \cdots}\rangle = 0 \; . \qquad (3.2)$$

Since these states are protected by supersymmetry, we expect this to be true independently of the gauge coupling.

From eqs. (3.1) and (3.2) it is natural to guess that the effective dynamics of these states will be described by a normal gauged matrix model with an harmonic oscillator potential [12, 30, 32, 33],

$$S = \int dt \; \text{Tr}(|D_t Z|^2 - |Z|^2) \; , \quad [Z, \bar{Z}] = 0 \; . \qquad (3.3)$$

This model can be visualized as a reduction of SYM on $\mathbb{R} \times S^3$ down to the zero mode of a single scalar [17, 12]. The eigenstates of the matrix model can be expressed as antisymmetric wave function of the complex eigenvalues z_i and can be classified in terms of Young Tableux [12]. Their quantum numbers match (3.1).

Moreover, in the large N limit generic coherent states are described by "droplet"-like distributions of eigenvalues on the complex plane. For example, the ground state $\psi_0 \sim e^{-\text{Tr}(Z\bar{Z})}$ will have a probability density,

$$\langle \psi_0 | \psi_0 \rangle = \int_{[Z, \bar{Z}]=0} [dZ d\bar{Z}] |\psi_0|^2 \propto \int \prod_{i=1}^{N} d^2 z_i \; e^{-2 \sum_i |z_i|^2 + \sum_{i<j} \log |z_i - z_j|^2} \; , \qquad (3.4)$$

which will be dominated by the saddle point of the exponential in the large N limit. Here we have used the measure change for a normal matrix model which follows from the metric $ds^2 = \text{Tr}(dZ d\bar{Z})$ [34].

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[3]In this paper we will make heavy use of the operator/state correspondence of SYM. We will go back and forth between operators and states and we hope that the context will make clear which one we are using.

In the limit $N \to \infty$ one replaces the sums by density distributions and one extremizes the functional

$$E[\rho] = -2 \int d^2z \rho(z) |z|^2 + \frac{1}{2} \int \int d^2 z_1 d^2 z_2 \rho(z_1) \rho(z_2) \log |z_1 - z_2|^2 \ , \qquad (3.5)$$

with the constraint $N = \int d^2 z \rho(z)$. Using the fact that the logarithm is the Green's function in two dimensions one obtains a circular droplet distribution of eigenvalues of constant density given by [34]

$$\sigma = -\frac{\Delta W(z, \bar{z})}{4\pi} = -\frac{\partial \bar{\partial} W(z, \bar{z})}{\pi} = \frac{2}{\pi} \ , \qquad (3.6)$$

where $W(z, \bar{z}) = -2|z|^2$ is the potential for the eigenvalues. Then, from the normalization of the density one obtains the radius of the droplet: $r_0 = \sqrt{N/2}$. The droplet approximation is very useful for calculating correlation functions in position space as we will see below.

3.2 SU(2) states dynamics

A generic SU(2) operator has the form:

$$\text{Tr}(ZYZZ \cdots) \, \text{Tr}(ZYYZ \cdots) \cdots \ . \qquad (3.7)$$

For these holomorphic states one still has that the D-term contributions add to zero at one loop. Therefore, we expect that we can also ignore the D-terms from the effective matrix model. The lack of supersymmetry can make this model much more complicated that the 1/2 BPS case. However one can argue along the lines of [17] that for the 1/4 BPS states the corresponding matrix model is a simple rotation of (3.3):

$$S_{1/4BPS} = \sum_{\alpha=Y,Z} \text{Tr} \left(|D_t Z_\alpha|^2 - |Z_\alpha|^2 \right) \ , \qquad (3.8)$$

for normal commuting matrices $[Z_\alpha, Z_\beta] = [Z_\alpha, \bar{Z}_\alpha] = 0$.

At one loop, the anomalous dimension is again generated by the commutators from the F-terms: $\text{Tr}|[Z_\alpha, Z_\beta]|^2$. We can again argue for an all-loop generalization [21] involving two *complex* matrices and ignoring the D-terms:

$$S = \int dt \, \text{Tr} \left[\sum_\alpha (|D_t Z_\alpha|^2 - |Z_\alpha|^2) - \frac{g_{\text{YM}}^2}{(2\pi)^2} \sum_{\alpha,\beta} |[Z_\alpha, Z_\beta]|^2 + \text{higher commutators} \right] . \quad (3.9)$$

The first three terms of the action come from the direct reduction of SYM on the S^3 to the zero mode of the matrices, and the higher commutators will come from integrating out higher modes and fields.[4] A similar matrix model arises from the one-loop dilatation operator in the SU(2) sector [35, 36], however the canonical structure is very different.

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[4]Here the SO(6) invariant potential of SYM can be written as $\sum_{a,b=1}^6 \text{Tr}|[X_a, X_b]|^2 = 2\sum_{\alpha,\beta=1}^3 \text{Tr}|[Z_\alpha, Z_\beta]|^2 + 2\sum_{\alpha,\beta=1}^3 \text{Tr}|[Z_\alpha, \bar{Z}_\beta]|^2$, where $Z = \frac{1}{\sqrt{2}}(X_1 + iX_2)$ etc. The last term in the potential is the D-term that we ignore in this matrix model.

As in any quantum system one chooses a particular classical configuration for which to define the perturbation theory. To define a consistent perturbative expansion one needs to find stable classical configurations. In our case, we know that in the large N limit (and with $Y = 0$) the normal matrix model $[Z, \bar{Z}] = 0$ can be described by droplets on the single particle phase space of the eigenvalues of Z. Moreover, these configurations are stabilized by SUSY. It then makes sense to expand around these "classical" solutions. Since the classicality is only statistical, one needs a prescription to define this expansion. We do this in the following way. First write as usual $Z \rightarrow Z + \delta Z$, and we diagonalize the background Z. Then, we treat the eigenvalues z_i as random numbers with probability distribution $\sim \exp(-2 \sum_i |z_i|^2 + \sum_{i<j} \log |z_i - z_j|^2)$. The resulting operators and states of the Hilbert space will depend on z_i. Therefore, we define the inner product to be the statistical average of the usual one:

$$\langle \phi | \hat{\mathcal{O}} | \tilde{\phi} \rangle \equiv \frac{\int \prod_i d^2 z_i |\psi_0(\{z_i\})|^2 \langle \phi(\{z_i\}) | \hat{\mathcal{O}}(\{z_i\}) | \tilde{\phi}(\{z_i\}) \rangle}{\int \prod_i d^2 z_i |\psi_0(\{z_i\})|^2} \ . \tag{3.10}$$

In the large N limit we can use the saddle point approximation as before.

At this point, our procedure is pretty much equivalent to the one in [37, 38], but the approach presented here is simpler and allows generalizations beyond the SU(2) sector as we will see in the next sections.

3.3 1/2 BPS condensates and string bits

We can now calculate the effective hamiltonian for the Y fields on the background of Z. As we discussed before, we expand $Z \rightarrow Z + \delta Z$. It is easy to see from the action (3.9) that the backreaction δZ enters quadratically into the action. Therefore, we can integrate it out and get an effective action for the Y field only. This will produce higher order interactions for the Y field. At this point, we will ignore these interactions. We will come back to them in section 5.

The remaining part of the action (3.9) is,

$$S = \int dt \sum_{i,j} \left[\dot{Y}_i^j (\dot{\overline{Y}})_j^i - w_{ij}^2 Y_i^j (\overline{Y})_j^i \right] \ . \tag{3.11}$$

where we have taken the $A_0 = 0$ gauge, the dots indicate time derivatives and,

$$\omega_{ij} = \sqrt{1 + \frac{\lambda}{(2\pi)^2} |z_i - z_j|^2} \ , \tag{3.12}$$

with z_i the eigenvalues of the background Z matrix. These eigenvalues will be determined using the statistical weight defined in the previous section. Here we have normalized the diagonal matrix so that $z_i = r_i e^{i\phi_i}$ with $r_i \leq 1$.

The Hamiltonian that follows from the action (3.11) can be written as,

$$H^{(2)} = \sum_{i,j} \omega_{ij} \left[(A_Y^\dagger)_i^j (A_Y)_j^i + (A_{\bar{Y}}^\dagger)_i^j (A_{\bar{Y}})_j^i \right] \ , \tag{3.13}$$

where the creation operators are given by

$$(A_Y^\dagger)_i^j = \sqrt{\frac{w_{ij}}{2}}\left[Y_i^j - \frac{i}{\sqrt{w_{ij}}}(\pi_Y)_i^j\right] , \tag{3.14}$$

$$(A_{\bar Y}^\dagger)_i^j = \sqrt{\frac{w_{ij}}{2}}\left[\bar Y_i^j - \frac{i}{\sqrt{w_{ij}}}(\pi_{\bar Y})_i^j\right] . \tag{3.15}$$

Here π_Y and $\pi_{\bar Y}$ are the canonical momenta to the coordinates Y and $\bar Y$ respectively. They obey the usual canonical commutator relations,

$$[Y_i^j, (\pi_Y)_k^l] = i\delta_i^l\delta_k^j , \quad [\bar Y_i^j, (\pi_{\bar Y})_k^l] = i\delta_i^l\delta_k^j , \tag{3.16}$$

with all other commutators equal to zero. One can then easily derive the standard oscillator commutator relations,

$$[(A_Y)_j^i, (A_Y^\dagger)_k^l] = \delta_i^l\delta_k^j , \quad [(A_{\bar Y})_j^i, (A_{\bar Y}^\dagger)_k^l] = \delta_i^l\delta_k^j , \tag{3.17}$$

with all other commutators equal to zero. Note that we have assumed that the fermions cancel the zero-point energy from normal ordering the operators in (3.13).

We can now construct the Hilbert space of states. With our normalization a generic single trace state takes the form

$$|n_1, n_2, \ldots, n_L\rangle = \frac{1}{N^{L/2}} \, \text{Tr}(A_Y^\dagger \psi_{n_1}(Z) A_Y^\dagger \psi_{n_2}(Z) \cdots A_Y^\dagger \psi_{n_L}(Z))|0\rangle_Y , \tag{3.18}$$

where,

$$\psi_n(Z) = \sqrt{1+n}Z^n , \tag{3.19}$$

and $|0\rangle_Y$ is the usual vacuum for Y defined by $A_Y|0\rangle_Y = A_{\bar Y}|0\rangle_Y = 0$. The wavefunctions $\psi_n(Z)$ are dual to the 1/2 BPS condensates. In fact as we will see they will localize on an S^1 at large n just like the 1/2 BPS states. This is the infinite momentum limit. The $A_Y^\dagger$ excitations are called "string bits" [17, 21] and as we will see they have a dual description as the "giant magnons" of [22].

It is easy to verify the orthonormality of these states using our inner product prescription:

$$\langle n_1, n_2, \ldots, n_L|n_1', n_2', \ldots, n_L'\rangle \approx \prod_{l=1}^L \int[dD_l]\psi_{n_l}(z_l)^*\psi_{n_l'}(z_l) = \prod_{l=1}^L \delta_{n_l, n_l'} , \tag{3.20}$$

where $\int[dD] = \int_0^1 dr\, r \int_0^{2\pi} d\phi/\pi$ is the integration across the droplet, and we are assuming the generic case where not all of the n_i are equal so we ignore the cyclicity of the trace.

Now lets consider calculating the expectation value of some observable, say the Hamiltonian (3.13). After doing the usual planar contractions of the A_Ys one can always reduce the problem to a product of integrals over the droplet. A useful property of the 1/2 BPS condensates is

$$Z\psi_n(Z) = \sqrt{\frac{n+1}{n+2}}\psi_{n+1}(Z) . \tag{3.21}$$

Under the inner product (3.20) one then sees that Z and $\bar{Z}$ can be treated as the operators $\hat{Z}^\dagger$ and $\hat{Z}$ with the property:

$$\hat{Z}^\dagger|n\rangle = \sqrt{\frac{n+1}{n+2}}|n+1\rangle \equiv \hat{r}e^{i\hat{\phi}}|n\rangle \tag{3.22}$$

$$\hat{Z}|n\rangle = \sqrt{\frac{n}{n+1}}|n-1\rangle \equiv e^{-i\hat{\phi}}\hat{r}|n\rangle\,, \tag{3.23}$$

$$\tag{3.24}$$

where we define

$$\hat{r} = \sqrt{\frac{\hat{n}}{\hat{n}+1}}\,, \quad e^{i\hat{\phi}}|n\rangle = |n+1\rangle\,. \tag{3.25}$$

Note that $\langle\hat{r}\rangle \leq 1$ and thus as the notation suggests, this will become the operator that measures the radial distance of the droplet. We also observe that $\hat{n}$ is the momentum conjugate to $\hat{\phi}$, $[\hat{\phi}, \hat{n}] = i$, or in terms of $\hat{r}$, $\hat{p}_\phi = \hat{r}^2/(1-\hat{r}^2)$. From the canonical commutator relation we can derive,

$$[\hat{\phi}, \hat{r}] = i\frac{(1-\hat{r}^2)^2}{2\hat{r}}\,. \tag{3.26}$$

This indicates that our system is constrained, which is not surprising since we are restricting our Hilbert space by choosing these special SU(2) states. As we will see, this is exactly the expected canonical structure for the SU(2) states in the string theory dual after an appropriate gauge choice.

It is easy to show that for any function $f(Z, \bar{Z})$ with a power law expansion,

$$\int[dD]\psi_n(z)^* f(z, \bar{z})\psi_{n'}(z) \cong \langle n|{}^\circ_\circ f(\hat{Z}^\dagger, \hat{Z}){}^\circ_\circ|n'\rangle\,, \tag{3.27}$$

where ${}^{\circ\circ}_{\circ\circ}$ denotes anti-normal ordering with respect to the operators $\hat{Z}^\dagger$, $\hat{Z}$. Therefore, we can write the quadratic Hamiltonian in our basis as

$$H^{(2)} = \sum_{l=1}^{L} {}^\circ_\circ \sqrt{1 + \frac{\lambda}{(2\pi)^2}|\hat{Z}_l^\dagger - \hat{Z}_{l+1}^\dagger|^2} \, {}^\circ_\circ\,. \tag{3.28}$$

Note that in contrast to [21], here we have given a quite explicit form of the Hamiltonian for the "string bits". The claim is that this "simple" Hamiltonian describes the SU(2) sector of the gauge theory at *strong coupling*. However, as we will discuss in the next section, we expect this Hamiltonian to be valid only to leading order in the $n_i \to \infty$ limit.

Moreover, note that the procedure presented in this section can be easily extended to an arbitrary 1/2 BPS droplet. The only difference is that one needs to change the integration $\int[dD]$ to the corresponding integration over the deformed droplet. Moreover, the orthogonal wavefunctions $\psi_n(Z)$ will no longer have the simple form Z^n. It would be interesting to study Giant Magnons in generic 1/2 BPS geometries using this formalism.
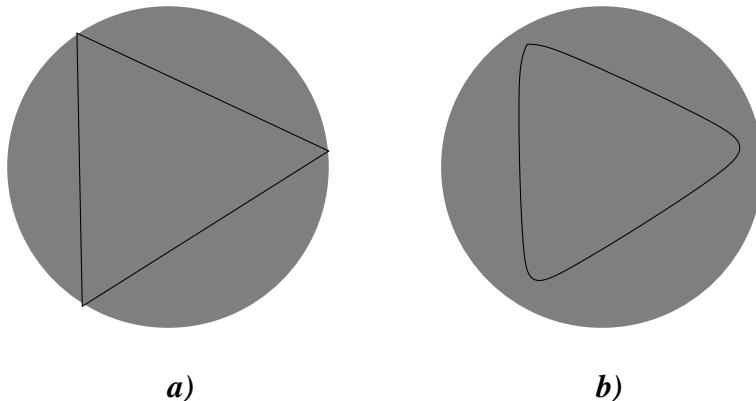
Figure 1: In figure a) we show the usual giant magnons at infinite momenta $n_i \to \infty$. The lines are identified with the "string bits" Y_i^j and the edges with the 1/2 BPS condensates Z^{n_i}. Here the disk is mapped to an $S^2 \subset S^5$ so that the edge is the equator of the S^2. Finite n_i effects will delocalize the ends of the magnons away from the equator as shown in figure b).

3.4 Classical limit and the dual string theory

In Quantum Mechanics one usually recovers the classical limit by taking $\hbar \to 0$. This makes the canonical commutators vanish, $[x, p] = i\hbar \to 0$ so that x and p become simple classical observables. For constrained systems the classical limit can be trickier. In our case we can see from (3.26) that the classical limit is reached by taking states for which $\langle \hat{r} \rangle \to 1$, or $\langle \hat{n} \rangle \to \infty$.[5] This is precisely the localization on the edge of the droplet which is correlated to the limit where the Y impurities are "far away". This limit also takes away the ordering ambiguities from the Hamiltonian which becomes

$$H^{(2)} \approx \sum_{l=1}^{L} \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{\phi_l - \phi_{l+1}}{2} \right) , \qquad (3.29)$$

where we have taken the large λ limit to compare with the string theory.

This is precisely the Hamiltonian of the "giant magnons" of [22]. In fact, the whole picture is exactly the same: we can picture the Y_i^j as an excitation (string bit) joining two eigenvalues z_i and z_j which become localized at the edge of the droplet when $n \to \infty$. In fact, taking $n \to \infty$ with $\Delta\phi$ fixed is precisely the Hoffman-Maldacena limit. Here, however, we have a full quantum description of the system. One can see that finite n effects will delocalize the ends of the string bits and make them "fuzzy" on the S^2 (figure 1). Finite n effects were considered recently in [39] for a single giant magnon. Their results confirm that the ends of the giant magnon are delocalized from the equator at finite n.

The relation between the momentum of the magnons and the angles on the droplet can be seen more clearly by Fourier transforming as in [21]. For example, consider the following asymptotic state,

$$|\psi(p)\rangle = \lim_{\Delta \to \infty} \lim_{n \to \infty} \sum_{x=-\Delta}^{\Delta} e^{ipx} \, \text{Tr}(\cdots Y^\dagger Z^{n+x} Y^\dagger Z^{n-x} Y^\dagger \cdots)|0\rangle_{\text{Y}} , \qquad (3.30)$$

---

[5]Note that in our conventions $\hbar = 1$.

where we first take the limit of infinite separation $n \to \infty$. One can show that these states are approximately orthogonal in this limit.

For condensates of infinite angular momentum the operators (3.22) and (3.23) become simple shift generators,

$$\hat{Z}|n\rangle \approx |n-1\rangle \ , \quad \hat{Z}^\dagger|n\rangle \approx |n+1\rangle \ . \tag{3.31}$$

Focusing on a single string bit, the effective Hamiltonian between these asymptotic states is simply,

$$H^{(2)}_{l,l+1} \approx \sqrt{1 + \frac{\lambda}{(2\pi)^2}\mathcal{H}_{l,l+1}} \ , \tag{3.32}$$

where,

$$\mathcal{H}_{l,l+1}|n_l, n_{l+1}\rangle = 2|n_l, n_{l+1}\rangle - |n_l - 1, n_{l+1} + 1\rangle - |n_l + 1, n_{l+1} - 1\rangle \ . \tag{3.33}$$

This is the same asymptotic Hamiltonian found in the two matrix model of [38]. The eigenstates of this Hamiltonian are simply plane waves of the form (3.30) with energy,

$$E_{\text{asymp.}} = \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right)} \ . \tag{3.34}$$

Comparing with (3.29) we see that $p \cong \Delta\phi$. Note that (3.34) is exactly the energy of the Giant Magnons at strong coupling. Moreover, it gives the correct *all loop* dispersion relation of the magnons that follow from the Bethe Ansatz [8].

We can also match the canonical structure of these string bits to the one found in the string theory side. Since these are SU(2) states they must be a limit of the well known rotating strings on $\mathbb{R} \times S^3$ [5]. A well known limit of these strings is the one corresponding to "long strings": $L \to \infty$ with $\lambda/L^2 = $ fixed $\ll 1$. The canonical structure should not depend on the particular limit we are taking since L is a conserved quantum number. One can calculate the Polyakov action for these string using the following coordinates on the S^3 [24]:

$$Z = re^{i(t+\phi)} \ , \quad Y = \sqrt{1-r^2}e^{i\varphi} \ . \tag{3.35}$$

One then chooses the gauge, $\tau = t$, $p_\varphi = $ const., which is appropriate to compare with the bosonized labeling of the states (3.18). This type of gauge was introduced in [40, 41]. One obtains the action,[6]

$$S \approx L \int dt \int_0^1 d\sigma \left[\frac{r^2\dot{\phi}}{1-r^2} - 1 - \frac{\lambda}{8\pi^2 L^2}(r'^2 + r^2\phi'^2) + \mathcal{O}\left(\frac{\lambda^2}{L^4}\right) \right] \ , \tag{3.36}$$

where one eliminates the time derivatives of all the higher order terms [40, 41] and,

$$L = \sqrt{\lambda} \int_0^1 d\sigma p_\varphi \ , \quad \lambda = g^2_{\text{YM}}N = R^4/\alpha'^2 \ . \tag{3.37}$$

---

[6]In [24] we dropped the factor of -1 since we were comparing anomalous dimensions only.

The same action can be found from the spin chain formalism in the gauge theory at one loop using the coherent states for the Cuntz algebra [24]. From this action we see that the canonical momenta are

$$p_r = 0 \ , \quad p_\phi = L\frac{r^2}{1 - r^2} \ . \tag{3.38}$$

This are precisely the constraints found above from the matrix model calculation!

The additional factor of L comes from taking the large L limit, since the total momentum for the Z fields in the gauge theory becomes:

$$\hat{J}_Z = \sum_{l=1}^{L} \hat{n}_l \approx L \int_0^1 d\sigma \frac{\hat{r}^2}{1 - \hat{r}^2} \ . \tag{3.39}$$

One can also reproduce the commutator algebra by using the classical Dirac brackets for constrained systems [42]. The Dirac bracket is defined by

$$\{A, B\}_{\mathrm{D}} = \{A, B\}_{\mathrm{PB}} - \sum_{i,j}\{A, f_i\}_{\mathrm{PB}}(G^{-1})_{ij}\{f_j, B\}_{\mathrm{PB}} \ , \tag{3.40}$$

where $\{,\}_{\mathrm{PB}}$ is the usual Poisson bracket. Furthermore, the second class constraints are given by the equations $f_i = 0$ and G^{-1} is the inverse of

$$G_{ij} = \{f_i, f_j\}_{\mathrm{PB}} \ . \tag{3.41}$$

The constraints are given by:

$$f_1 = p_r \ , \quad f_2 = p_\phi - L\frac{r^2}{1 - r^2} \ . \tag{3.42}$$

Then it is straightforward to verify

$$\{\phi, r\}_{\mathrm{D}} = \frac{(1 - r^2)^2}{2Lr} \ . \tag{3.43}$$

The comparison with the quantum theory is done as usual: $[,] = i\{,\}_{\mathrm{class.}}$. One then obtains precisely the continuum version of the commutators (3.26).

We now recall that in the last section we ignored the higher order interactions that come from integrating out the backreaction δZ from the action (3.9). This amounts to ignoring the backreaction to the 1/2 BPS condensates all together. As we have seen in this section, the condensates behave classically only in the $n_i \to \infty$ limit. Therefore, we expect that the quantum backreaction to the condensates will come as a $1/n_i$ effect. It would be interesting to confirm this by doing a direct matrix model calculation. This is, however, outside the scope of this paper.

3.5 Multiple giant magnons and bound states

Suppose that we put many string bits together as in the state (2.1). One of two things can happen: either we have the trivial addition of classical giant magnons, or we form a bound

state. The first outcome happens if we take the formal $n_i \to \infty$ limit for each condensate. The total energy in the limit of many string bits ($L \gg 1$) is,

$$H^{(2)} \approx \sum_{l=1}^{L} \frac{\sqrt{\lambda}}{\pi} \sin\left(\frac{\phi_l - \phi_{l+1}}{2}\right) \approx \frac{\sqrt{\lambda}}{2\pi} \int_0^1 d\sigma \partial_\sigma \phi = \frac{\sqrt{\lambda}}{2\pi} \Delta\phi \,. \qquad (3.44)$$

Here we assume that L does not scale in any way with λ. If we scale L as in the fast string limit, $\lambda/L^2 \ll 1$ we get instead,

$$H^{(2)} \approx L \int_0^1 d\sigma \left(1 + \frac{\lambda}{8\pi^2 L^2} \phi'^2 + \cdots\right) \,. \qquad (3.45)$$

This agrees with the classical string action (3.36) at one loop.

Understanding the emergence of bound states and of strings at $r < 1$, requires taking into account $1/n_i$ corrections. However, there are two effects that can be important for understanding these corrections. First, as we discussed above we need to take into account the backreaction term in the matrix model. Furthermore, one needs to understand the possible higher order interactions that can come from integrating higher spherical harmonics in SYM. We do not have a good understanding of these issues at this moment. However, suppose that these unknown interactions tend to normal order the Hamiltonian (3.28). Then one can deduce the classical limit by constructing coherent states for the operators $\hat{Z}_l$.

In fact, one can show that the states,

$$|z\rangle = (1 - |z|^2) \sum_{n=0}^{\infty} \sqrt{n+1} z^n |n\rangle \,, \qquad (3.46)$$

are indeed overcomplete coherent states of the operator $\hat{Z}$. The completeness relation is,

$$\int_0^1 \frac{dr\, r}{(1-r^2)^2} \int_0^{2\pi} \frac{d\phi}{\pi} |z\rangle\langle z| = 1 \,. \qquad (3.47)$$

The classical Hamiltonian in the coherent state basis will be,

$$\langle\, : H^{(2)} : \,\rangle = \sum_{l=1}^{L} \sqrt{1 + \frac{\lambda}{(2\pi)^2} |Z_l - Z_{l+1}|^2} \approx \frac{\sqrt{\lambda}}{2\pi} \int_0^1 \sqrt{r'^2 + r^2\phi'^2} \,, \qquad (3.48)$$

where in the last step we have taken the strong coupling limit and then the continuum limit corresponding to a large number of string bits. This is exactly the Nambu-Goto action for a string on $\mathbb{R} \times S^2$ in the static gauge [22].

Of course, the presence of bound states follows directly from the duality between this action and the Sine-Gordon theory [22]. These bound states and its generalizations have been studied recently in [27 – 29, 43 – 45].

4. 1/4 and 1/8 BPS condensates: the SU(3) sector

The concept of 1/4 BPS condensates was discussed briefly in section 2. Here we will study these condensates in detail as we did for the 1/2 BPS case. The general states we want to consider are of the form

$$\text{Tr}(X\{Z^{n_1}Y^{m_1}\}Y\{Z^{n_2}Y^{m_2}\}Z\cdots) . \tag{4.1}$$

In particular we are interested in the limit where $n_i, m_i \to \infty$ with the number of "impurities" outside of the condensates held fixed.

By the same arguments of the previous section, we expect an effective description of generic SU(3) states in terms of a matrix model similar to (3.9) but now with three complex matrices ($\alpha = X, Y, Z$). There are two complications in the SU(3) case, however. The fluctuations δX, δY and δZ cannot be integrated out so easily. This is actually a good thing as these fluctuations have an interpretation in the operator language as we discussed in section 2.

Moreover, it turns out that we need to include the D-terms to the action. This conclusion follows from comparing with the string theory results (see below). We do not have a purely field theoretical explanation for this but it seems to be a consequence of the fact that we have less supersymmetry for these states and hence the one loop result is not protected at strong coupling. The effective action for these states is then of the form,

$$S = \int dt \, \text{Tr}\Bigg[\sum_\alpha (|D_t Z_\alpha|^2 - |Z_\alpha|^2) - \frac{g_{\text{YM}}^2}{(2\pi)^2}\sum_{\alpha,\beta}|[Z_\alpha, Z_\beta]|^2$$
$$-\frac{g_{\text{YM}}^2}{(2\pi)^2}\sum_{\alpha,\beta}|[Z_\alpha, \bar{Z}_\beta]|^2 + \text{higher commutators}\Bigg] . \tag{4.2}$$

As we discussed in section 2, for large n_i, m_i the states (4.1) look like many "condensates" of 1/4 BPS operators involving Z and Y fields. Therefore we can try to define our expansion around "classical" configurations with $[Z, Y] = [Z, \bar{Z}] = [Y, \bar{Y}] = 0$. The classical configurations will be the eigenstates of the 1/4 BPS action (3.8) in the large N limit. In particular, we know that the ground state is given by $\psi_0 \sim e^{-\text{Tr}(|Z|^2 + |Y|^2)}$ [17]. Thus, the energy functional that determines the geometry of the eigenvalue distribution is

$$E[\rho] = -2\int d^4x \rho(x)(|z|^2 + |y|^2) + \frac{1}{2}\int\int d^4x_1 d^4x_2 \rho(x_1)\rho(x_2)\log(|y_1 - y_2|^2 + |z_1 - z_2|^2) , \tag{4.3}$$

where we have used the measure change for two normal commuting matrices:

$$[dZd\bar{Z}dYd\bar{Y}] \propto \prod_i d^2z_i d^2y_i \prod_{k<l}(|y_k - y_l|^2 + |z_k - z_l|^2) . \tag{4.4}$$

Doing a saddle point calculation as in [21] one finds that the eigenvalues form a singular distribution on an $S^3 \subset \mathbb{R}^4$ with radius $r_0 = \sqrt{N}/2$. The fluctuations around this background are the non-BPS parts of these operators.

The motivation for this interpretation is the same as with the 1/2 BPS condensates. If we consider calculating the one loop anomalous dimension for these states we see that the planar contribution from the F-term and D-term commutators $\mathrm{Tr}|[Z,Y]|^2$, $\mathrm{Tr}|[Z,\bar{Y}]|^2$ will be zero when contracted between the condensates. Only contractions between fields outside the condensates and the condensates themselves will give rise to anomalous dimensions. We interpret these interactions as fluctuations around the commuting background. The interactions $\mathrm{Tr}|[Z,X]|^2$, $\mathrm{Tr}|[Z,\bar{X}]|^2$ and $\mathrm{Tr}|[Y,X]|^2$, $\mathrm{Tr}|[Y,\bar{X}]|^2$ that involve a condensate and a transverse X field are interpreted as the fluctuation δX in the matrix model. Finally the interactions $\mathrm{Tr}|[Z,Y]|^2$, $\mathrm{Tr}|[Z,\bar{Y}]|^2$ with Z or Y fields outside the condensate are just the backreaction of the condensate δY and δZ.

4.1 Hilbert space and canonical structure

To simplify the discussion let us start with the states

$$\mathrm{Tr}(X\{Z^{n_1}Y^{m_1}\}X\{Z^{n_2}Y^{m_2}\}X\cdots) . \tag{4.5}$$

That is, let us ignore the backreaction to the 1/4 BPS condensates for the moment ($\delta Y = \delta Z = 0$). We then diagonalize the classical background of commuting normal matrices. The effective Hamiltonian for the X fluctuations will have the same form as for the SU(2) case (3.13) but now with the following dispersion relation

$$\omega_{ij} = \sqrt{1 + \frac{\lambda}{(2\pi)^2}\left(|y_i - y_j|^2 + |z_i - z_j|^2\right)} . \tag{4.6}$$

Note that if we had ignored the D-terms, the dispersion relation would have an additional factor of 1/2: $\sqrt{1 + \frac{\lambda}{2(2\pi)^2}(|\Delta z|^2 + |\Delta y|^2)}$.

In analogy with the 1/2 BPS condensates, the basis for this sector is

$$|n_1,m_1;n_2,m_2;\ldots;n_L,m_L\rangle = \frac{1}{N^{L/2}}\,\mathrm{Tr}[A_X^\dagger \psi_{n_1,m_1}(Y,Z)\cdots A_X^\dagger \psi_{n_L,m_L}(Y,Z)]|0\rangle_X , \tag{4.7}$$

with

$$\psi_{n,m}(Y,Z) = \sqrt{\frac{(n+m+1)!}{n!m!}}Z^nY^m . \tag{4.8}$$

Using the saddle point approximation, the inner product can be reduced to integrals over an S^3 in the large N limit:

$$\langle n_1,m_1;\ldots;n_L,m_L|n_1',m_1';\ldots;n_L',m_L'\rangle \approx \prod_{l=1}^{L}\int \frac{(d\Omega_3)_l}{\mathrm{Vol}(S^3)}\psi_{n_l,m_l}(y_l,z_l)^*\psi_{n_l',m_l'}(y_l,z_l)$$

$$= \prod_{l=1}^{L}\delta_{n_l,n_l'}\delta_{m_l,m_l'} , \tag{4.9}$$

where z,y become coordinates on the three-sphere, $|y|^2 + |z|^2 = 1$.

Just like in the SU(2) sector, we can treat Z and Y as creation operators under our inner product. They obey the following properties:

$$\hat{Z}^{\dagger}|\psi_{n,m}\rangle = \sqrt{\frac{n+1}{2+n+m}}|\psi_{n+1,m}\rangle \equiv \frac{\hat{r}_1}{\sqrt{1+\hat{\eta}}}e^{i\hat{\phi}_1}|\psi_{n,m}\rangle \, , \tag{4.10}$$

$$\hat{Y}^{\dagger}|\psi_{n,m}\rangle = \sqrt{\frac{m+1}{2+n+m}}|\psi_{n,m+1}\rangle \equiv \frac{\hat{r}_2}{\sqrt{1+\hat{\eta}}}e^{i\hat{\phi}_2}|\psi_{n,m}\rangle \, , \tag{4.11}$$

$$e^{i\hat{\phi}_1}|\psi_{n,m}\rangle = |\psi_{n+1,m}\rangle \, , \tag{4.12}$$

$$e^{i\hat{\phi}_2}|\psi_{n,m}\rangle = |\psi_{n,m+1}\rangle \, , \tag{4.13}$$

$$\hat{r}_1 = \sqrt{\frac{\hat{n}}{\hat{n}+\hat{m}}} \, , \tag{4.14}$$

$$\hat{r}_2 = \sqrt{\frac{\hat{m}}{\hat{n}+\hat{m}}} \, , \tag{4.15}$$

$$\hat{r}_1^2 + \hat{r}_2^2 = 1 \, , \tag{4.16}$$

$$\hat{\eta} = \frac{1}{\hat{n}+\hat{m}} \, . \tag{4.17}$$

We can now calculate the following canonical structure:

$$[\hat{\phi}_1, \hat{r}_1] = i\frac{\hat{r}_2^2}{2\hat{r}_1}\hat{\eta} \, , \tag{4.18}$$

$$[\hat{\phi}_1, \hat{r}_2] = -i\frac{\hat{r}_2}{2}\hat{\eta} \, , \tag{4.19}$$

$$[\hat{\eta}, \hat{\phi}_\alpha] = i\hat{\eta}^2 \, , \tag{4.20}$$

and the conjugate momentum to $\hat{\phi}_\alpha$ is $\hat{P}_\alpha = \hat{r}_\alpha^2/\hat{\eta} = \hat{n}_\alpha$.

One can also show that the operators are antinormal ordered under the inner product (see appendix A) and so the effective quadratic Hamiltonian is just

$$H^{(2)} = \sum_{l=1}^{L} {}^{\circ}_{\circ}\sqrt{1 + \frac{\lambda}{(2\pi)^2}\left(|\hat{Y}_l^{\dagger} - \hat{Y}_{l+1}^{\dagger}|^2 + |\hat{Z}_l^{\dagger} - \hat{Z}_{l+1}^{\dagger}|^2\right)}{}^{\circ}_{\circ} \, . \tag{4.21}$$

4.2 Localization and the classical limit

The localization of the 1/4 BPS condensates works just like for the 1/2 BPS case. Looking at the canonical commutators (4.10) we see that for states with $\langle\hat{n}+\hat{m}\rangle \to \infty$ with $\langle\hat{n}\rangle/\langle\hat{m}\rangle$ fixed will localize on an S^3. The operators $\hat{r}_\alpha$, $\hat{\phi}_\alpha$ become commuting (classical) numbers and we can drop the anti-normal ordering symbols on the Hamiltonian (4.21) and replace the operators by classical coordinates on the S^3, $|Y_l|^2 + |Z_l|^2 = 1$.

In the case of the 1/2 BPS condensates, the localization occurs from $S^2 \to S^1$. The form of the commutators (4.10) suggests that for the 1/4 BPS case the localization occurs as $S^4 \to S^3$. We will confirm this intuition by comparing with the dual string states. But first lets try to match the canonical structure as we did with the 1/2 BPS condensates.

It is now important to remind the reader that the canonical structure found in the string theory side is sensitive to 1) the string configurations that we are considering and 2)

the gauge choice in the sigma model action. One could think that since the states (4.5) are a restricted subset of the SU(3) sector (ignoring backreaction) one will not be able to match the canonical structure found in the gauge theory side. However this is not correct since we can always expand around these states and, if we take into account the backreaction to the 1/4 BPS condensates, we have a complete basis of states for this sector.[7] Thus in this case the particular canonical structure is tied to the choice of basis and therefore, in the string theory side, it will be related to the gauge choice in the sigma model. We would have obtained a different canonical structure had we expanded around 1/8 BPS condensates which are of the form $\mathrm{Tr}(\{X^n Y^m Z^p\} \cdots)$ for example (see below).

We can now find the sigma model action for the SU(3) sector in the "fast string" limit: $J_X \to \infty$, $\lambda/J_X^2 = $ fixed, just like we did in the SU(2) sector. The form of the operators (4.5) tells us that the correct gauge choice is the one that distributes the angular momentum in X uniformly along the string. Looking at the commutators (4.10) one realizes that a convenient spacetime coordinate system for these strings is,

$$Z = \frac{r_1}{\sqrt{1+\eta}} e^{i(t+\phi_1)} , \quad Y = \frac{r_2}{\sqrt{1+\eta}} e^{i(t+\phi_2)} , \quad X = \sqrt{\frac{\eta}{1+\eta}} e^{i\varphi} , \qquad (4.22)$$

where $r_1^2 + r_2^2 = 1$. We now choose the gauge

$$t = \tau , \quad p_\varphi = \mathrm{const.} \qquad (4.23)$$

Following the standard procedure of eliminating time derivatives for spatial derivatives [40, 41] in the sigma model one finds (see appendix B),

$$S = L \int d\tau \int_0^1 d\sigma \left[\frac{r_1^2}{\eta} \dot{\phi}_1 + \frac{r_2^2}{\eta} \dot{\phi}_2 \right. \qquad (4.24)$$
$$\left. -1 - \frac{\lambda \eta}{2(4\pi)^2 L^2 (1+\eta)^2} \left(\frac{\eta'^2}{4\eta} + \sum_{\alpha=1,2} (r_\alpha'^2 + r_\alpha^2 \phi_\alpha'^2) \right) + \mathcal{O}\left(\frac{\lambda^2}{L^4} \right) \right] .$$

We see that the canonical structure is exactly as in the matrix model calculation: $p_{\phi_\alpha} \sim r_\alpha^2/\eta$ with the other momenta set to zero. In fact, one can confirm the continuum version of the commutators (4.18) - (4.20) by using the Dirac brackets with the following constraints:

$$f_1 = r_1^2 + r_2^2 - 1 , \quad f_2 = p_{r_1} , \quad f_3 = p_{r_2} , \quad f_5 = p_{\phi_1} - Lr_1^2/\eta , \quad f_6 = p_{\phi_2} - Lr_2^2/\eta . \quad (4.25)$$

The localization on the S^3 can be understood in the same way as with the 1/2 BPS condensates: it correspond to the classical limit $\eta \to 0$.

Note however, that unlike in the SU(2) sector, the classical Hamiltonian that follows from the action (4.24) does not match with the naive classical limit of the matrix model Hamiltonian (4.21) at one loop. This is indeed not surprising since we are ignoring the backreaction to the condensates which is unavoidable in the SU(3) sector. We will come back to this point in section 5.

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[7]Note that even if we include the backreaction of the condensate, all terms in the Hamiltonian will be written in terms of the operators (4.10) and thus the canonical structure will be unchanged.

4.3 SU(3) giant magnons?

A natural question to ask at this point is whether we can match the matrix model Hamiltonian (4.21) at finite L with some sort of SU(3) giant magnon solution in the string theory side. In general one would expect that in the limit $n_l + m_l \to \infty$ each X string bit would correspond to a giant magnon configuration connecting two null geodesics of the form,

$$Z = r_1 e^{it} , \quad Y = r_2 e^{it} , \quad r_1^2 + r_2^2 = 1 , \qquad (4.26)$$

one for each condensate to the left and right of the string bit.[8]

Note however that these 1/4 BPS condensates do not correspond to the Giant Magnons with multiple angular momenta studied recently in the literature [43–45] which represent *bound states* in the operator language.

The states (4.5) are very restricted if we ignore the backreaction to the condensates. Therefore, we can only expect a matching for some very special configurations. In particular one intuitively expects that configurations for which the ends of the giant magnon are at different radii r_α are unstable and thus would require the understanding of the backreaction since on the dual gauge theory, one would have Z and Y fields flowing from one condensate to the next. We should then consider configurations for which r_α is the same at each site. In the dual string theory this would correspond to giant magnons connecting the same null trajectory.

The classical limit of the energy formula (4.21) for these special states is,

$$E_{\text{string bit}} = \frac{\sqrt{\lambda}}{\pi} \sqrt{r_1^2 \sin^2 \left(\frac{\Delta \phi_1}{2} \right) + r_2^2 \sin^2 \left(\frac{\Delta \phi_2}{2} \right)} . \qquad (4.27)$$

It turns out that with our simple ansatz for the classical string (see below), the matching with this quadratic formula only works for a subset of these states: those with $\Delta \phi_1 = \Delta \phi_2$. These can be considered as "rotations" of the usual SU(2) giant magnons. This restricted matching is hardly surprising since only in this case the spacetime probed by the string is actually flat. More generally, the backreaction to the string bit should take into account the curvature of the sphere and correct the naive square root form (4.27).

Let us now turn our attention to the classical string theory. We will consider strings moving on $\mathbb{R} \times S^4$ but for simplicity we restrict the motion on the $S^3 \subset S^4$ as follows:

$$Z = r_1 \sin[\psi(\sigma)] e^{i(\tau + \phi_1(\sigma))} , \quad Y = r_2 \sin[\psi(\sigma)] e^{i(\tau + \phi_2(\sigma))} , \quad X = \cos[\psi(\sigma)] . \qquad (4.28)$$

After defining new coordinates $\phi_1 = \phi_+ + \phi_-$ and $\phi_2 = \phi_+ - \phi_-$, the Nambu-Goto action in the static gauge $t = \tau$ becomes,

$$S_{\text{NG}} = \frac{\sqrt{\lambda}}{2\pi} \int d\tau d\phi_+ \sqrt{x'^2 + x^2 \left[1 + 2a\phi'_- + \phi'^2_-(1 - bx^2) \right]} , \qquad (4.29)$$

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[8]We can consider the more general null trajectories $Z = r_1 e^{i\omega_1 t}, Y = r_2 e^{i\omega_2 t}$ with $r_1^2 \omega_1^2 + r_2^2 \omega_2^2 = 1$, but this is the same as a redefinition of r_i. Therefore we will set $\omega_i = 1$ without loss of generality.

where $x = \sin\psi$, $a = r_1^2 - r_2^2$, $b = 4r_1^2 r_2^2$ and the derivative is with respect to ϕ_+. The equations of motion are,

$$\phi'_- = \frac{\alpha G - ax^2}{x^2(1 - bx^2)} \,, \tag{4.30}$$

$$u(x)G\frac{d}{dx}\left(\frac{u(x)}{G}\right) = x\left[1 + a\phi'_- + \phi'^2_-(1 - 2bx^2)\right] \,, \tag{4.31}$$

where

$$G = x\sqrt{\frac{a^2 x^2 - (u^2 + x^2)(1 - bx^2)}{\alpha^2 - x^2(1 - bx^2)}} \,, \tag{4.32}$$

$u = dx/d\phi_+$ and α is an integration constant. Therefore we can reduce the problem to a non-linear ODE for $u(x)$.

Let us now consider the special case of $r_1 = r_2 = 1/\sqrt{2}$. One can easily show that the reality of G implies that $\alpha = 0$ and so $\phi'_- = 0$. The equations of motion reduce exactly to the ones for the SU(2) giant magnon. The energy is [22],

$$E|_{r_1 = r_2} = \frac{\sqrt{\lambda}}{\pi}\left|\sin\left(\frac{\Delta\phi_+}{2}\right)\right| \,, \tag{4.33}$$

where $\Delta\phi_1 = \Delta\phi_2 = \Delta\phi_+$.

One can also show that setting $\alpha = 0$ implies $r_1 = r_2 = 1/\sqrt{2}$. This follows from the reality of $u(x)$. Setting $\alpha = 0$ and integrating the resulting equations of motion gives

$$u(x) = |b|\frac{x^2(1 - x^2)}{1 - bx^2}\sqrt{\frac{1 - bx_{\min}^2}{x_{\min}^2(1 - x_{\min}^2)} - \frac{1 - bx^2}{x^2(1 - x^2)}} \,, \tag{4.34}$$

where $x_{\min}$ is the turning point. At the boundary $x = 1$, the reality of $u(x)$ requires $b = 1$ which in turn implies $r_1 = r_2 = 1/\sqrt{2}$.

Now lets study the solutions with $\alpha \neq 0$. For these strings, the equation of motion for $u(x)$ is highly complicated. However it turns out that it has a very simple solution: the Giant Magnon of [22]. To see this, we first note that the boundary condition $u(x = 1)$ is actually determined by the EOM. When we set $x = 1$ in (4.31) all dependence on $u'(x)$ drops and one is left with an equation for $u(1)$:

$$u(1) = \frac{1}{\alpha}\sqrt{a^2 - \alpha^2} \,, \tag{4.35}$$

where we need $\text{sign}(\alpha) = \text{sign}(a)$, and $a^2 \geq \alpha^2$. This last inequality also follows from the reality of G at the boundary $x = 1$.

We can now relate this to the Giant Magnon solution in [22] for which,

$$u(x) = \frac{x^2}{x_{\min}}\sqrt{1 - \left(\frac{x_{\min}}{x}\right)^2} \,. \tag{4.36}$$

Comparing (4.36) with (4.35) we find that $\alpha = ax_{\min}$. With α now determined by the turning point $x_{\min}$ we can easily check that (4.36) satisfies (4.31). Therefore, as pointed

out before, the solutions that we have found are just rotations of the Giant Magnon of [22]. Nevertheless they are consistent with the interpretation in terms of 1/4 BPS condensates. This is because the ends of the Giant Magnon travel along null geodesics in S^3 which carry two angular momenta corresponding to the two fields Z and Y in the condensate, which is itself a rotation of the 1/2 BPS condensate.

The generalization to 1/8 BPS condensates should be obvious by now. These will be of the form $\mathrm{Tr}(\{X^n Y^m Z^p\} \cdots)$ but since there are no transverse excitations left, the inclusion of the backreaction is unavoidable. The eigenvalue distribution turns into a singular $S^5 \subset \mathbb{R}^6$ with radius $r_0 = \sqrt{N}/2$. In this case it is perhaps more useful to use a SO(6) invariant notation as in [21]. In this case, the dispersion relation in any direction $a = 1, \ldots, 6$ is,

$$w_{ij}^a = \sqrt{1 + \frac{\lambda}{(2\pi)^2} |\vec{x}_i - \vec{x}_j|^2} \,, \tag{4.37}$$

where $\vec{x}_i^2 = 1$ are the coordinates in the S^5.

5. Backreaction to BPS condensates

To simplify the discussion we can look at the 1/4 BPS condensates of the SU(2) sector. That is consider operators of the form

$$\mathrm{Tr}(Y\{Z^{n_1} Y^{m_1}\} Z \{Z^{n_2} Y^{m_2}\} \cdots) \,. \tag{5.1}$$

This way we can work with the simpler matrix model (3.9). The proposal is that we should expand around the "classical" configuration of commuting matrices $[Y, Z] = [Z, \bar{Z}] = [Y, \bar{Y}] = 0$ in the action (3.9) as we did for the SU(3) sector. Then the two possible excitations Y and Z outside the condensates will be described by the backreaction terms δY and δZ in the matrix model.

In this case it is difficult to make comparisons with the string theory dual because, as we will see, the number of excitations outside the condensates is not conserved. We can, however try to match the qualitative picture we expect from a formal field theory calculation using the operators (5.1). In the limit, $n_i, m_i \to \infty$ we again expect that the "impurities" outside the condensates will not interact with each other. On the other hand, we expect only interactions between impurities and the condensates. First, lets consider the quadratic fluctuation around the commuting background.

Expanding $Z_\alpha \to Z_\alpha + Y_\alpha$ where Z_α is the commuting background, one finds the following quadratic Hamiltonian (after diagonalizing the background)

$$H^{(2)} = \sum_{i,j} (\pi_\alpha)_i^j (\bar{\pi}_\alpha)_j^i + M_{ij}^{\alpha\beta} (Y_\alpha)_i^j (\bar{Y}_\beta)_j^i \,, \tag{5.2}$$

where,

$$M_{ij}^{\alpha\beta} = \begin{pmatrix} 1 + \alpha |z_{ij}|^2 & -\alpha z_{ij} \bar{y}_{ij} \\ -\alpha \bar{z}_{ij} y_{ij} & 1 + \alpha |y_{ij}|^2 \end{pmatrix} \tag{5.3}$$

and $\alpha = \lambda/(2\pi)^2$, $z_{ij} = z_i - z_j$ and similarly for y.

Naively one might think that the mass matrix can be diagonalized. However, we need to be very careful with the gauge invariance of the states. Diagonalizing the mass matrix means that we make a change of basis that depend on the background. On the other hand, any change of basis must be of the form,

$$(Y_\alpha)_i^j = \sum_\beta f_{ij}^\beta (\phi_\beta)_i^j \,, \tag{5.4}$$

where ϕ_β are the (normalized vectors) that diagonalize the mass matrix and, by gauge invariance, f_{ij}^β must only depend on positive powers of z_{ij} and y_{ij} (and their conjugates). Diagonalizing the mass matrix one finds the following eigenvectors and corresponding masses:

$$(\phi_1)_i^j = \frac{1}{\sqrt{|z_{ij}|^2 + |y_{ij}|^2}} \begin{pmatrix} z_{ij} \\ -y_{ij} \end{pmatrix}, \quad M_1^2 = 1 + \alpha(|z_{ij}|^2 + |y_{ij}|^2) \tag{5.5}$$

$$(\phi_2)_i^j = \frac{1}{\sqrt{|z_{ij}|^2 + |y_{ij}|^2}} \begin{pmatrix} \bar{y}_{ij} \\ \bar{z}_{ij} \end{pmatrix}, \quad M_2^2 = 1 \tag{5.6}$$

Inverting these relations one finds for example,

$$(Y_1)_i^j = \frac{1}{\sqrt{|z_{ij}|^2 + |y_{ij}|^2}} (\bar{z}_{ij}(\phi_1)_i^j + y_{ij}(\phi_2)_i^j) \,, \tag{5.7}$$

which is not allowed by gauge invariance. Even if we try to avoid this by not normalizing the eigenvectors one always runs into an ill defined square root at some point of the procedure (when defining the oscillator operators). This is telling us that $H^{(2)}$ is really an interacting Hamiltonian.

Using the usual oscillator basis we find $H^{(2)} = H_0^{(2)} + H_{int}^{(2)}$, where

$$H_0^{(2)} = w_{ij}^\alpha (A_\alpha^\dagger)_i^j (A_\alpha)_j^i \,, \tag{5.8}$$

$$H_{int}^{(2)} = -\frac{\alpha z_{ij} \bar{y}_{ij}}{2\sqrt{w_{ij}^y w_{ij}^z}} (A_y^\dagger)_i^j (A_z)_j^i + h.c. \,, \tag{5.9}$$

where $w_{ij}^y = \sqrt{1 + \alpha|z_{ij}|^2}$ etc. and we are taking expectation values on holomorphic states. Therefore, we observe that the interaction term represents the process of interchanging an "impurity" outside the condensate with one of the fields of the condensate (with opposite polarization):

$$\text{Tr}(\cdots Z\{Z^n Y^m\} \cdots) \leftrightarrow \text{Tr}(\cdots Y\{Z^{n+1} Y^{m-1}\} \cdots) \,. \tag{5.10}$$

There are also cubic and quartic interactions. Lets consider the cubic ones. These do not preserve the number of impurities. On holomorphic states these interactions take the following form:

$$H^{(3)} = \frac{\alpha z_{ij}}{r_0} (\tilde{A}_Y^\dagger)_i^j [\tilde{A}_Y, \tilde{A}_Z]_j^i + \frac{\alpha y_{ij}}{r_0} (\tilde{A}_Z^\dagger)_i^j [\tilde{A}_Z, \tilde{A}_Z]_j^i + h.c. \,, \tag{5.11}$$

where for simplicity of notation we defined the rescaled operators $(\tilde{A}_\alpha)_i^j = (A_\alpha)_i^j / \sqrt{2\omega_{ij}^\alpha}$. Furthermore, we have assumed normal ordering of the operators and the extra $1/r_0$ will be canceled since we loose/gain an extra field in the operator.

We see that these interactions involve the absorption/emission of one impurity from the condensate. For example,

$$\text{Tr}(\cdots Z\{Z^nY^m\}Y\cdots) \leftrightarrow \text{Tr}(\cdots Z\{Z^nY^{m+1}\}\cdots) . \tag{5.12}$$

Note, however, that the interaction involves *both* fields Z and Y at each side of the condensate. This prevents the creation of a single impurity out of the vacuum:

$$\text{Tr}(\{Z^nY^m\}) \to \text{Tr}(\{Z^nY^{m-1}\}Y) . \tag{5.13}$$

This matches our intuition from the Bethe Ansatz since there we must have zero total momentum along the trace (by cyclicity) and therefore we need at least two Bethe roots. Moreover, we expect that turning on a single commutator gives zero by the cyclicity of the trace:

$$\text{Tr}(\{Z^nY^m\}) \to \text{Tr}([\{Z^nY^{m-1}\},Y]) = 0 . \tag{5.14}$$

Finally we have the quartic vertex that involve the (long range) interaction between the two impurities at each side of a condensate:

$$H^{(4)} = 2\alpha \ \text{Tr}[\tilde{A}_Z^\dagger, \tilde{A}_Y^\dagger][\tilde{A}_Y, \tilde{A}_Z] , \tag{5.15}$$

where we have included the additional N that comes from the extra close loop in the Feynman diagram. Of course we expect higher commutators from integrating out the higher modes on the sphere.

It is easy to generalize this discussion to the SU(3) sector. For these operators we cannot avoid the inclusion of the backreaction and of the additional interactions from the D-terms. Therefore to fully understand the SU(3) sector we need to develop new techniques that can deal with lattices with varying number of sites. Note that this problem is already familiar in the study of Giant Gravitons [24] and multiple trace operators in SYM [46]. We do not know how to do this at this moment. But once this is understood we could calculate quantum corrections to the special states studied in the previous section.

5.1 Higher interactions and the strong coupling limit

Now that we have some experience in defining the expansion around commuting BPS condensates we would like to explain why this is in fact a strong coupling expansion and when does it breaks down. In other words, we want to understand under what circumstances, the quandratic (or cubic) approximation is a good one.

The secret to answer this question lies in the form of the creation/annihilation basis defined above. Note that the relation between the creation/annihilation operators and the matrix model coordinates is,

$$(Y_\alpha)_i^j = \frac{1}{\sqrt{2w_{ij}^\alpha}}[(A_\alpha^\dagger)_i^j + (\bar{A}_\alpha)_i^j] \sim \mathcal{O}\left(\frac{1}{(1+\lambda|\vec{x}_i - \vec{x}_j|^2)^{1/4}}\right) . \tag{5.16}$$

Therefore, if we consider excitations that join two eigenvalues whose distance $|\vec{x}_i - \vec{x}_j|^2$ is fixed in the limit $\lambda \to \infty$, then all interactions involving higher powers of the matrix

model coordinates will be naturally suppressed as,

$$\text{Tr}(Y^n) \sim \frac{1}{\lambda^{n/4}} \ . \tag{5.17}$$

Of course, we only expect higher commutators so we do not correct the quadratic potential. Moreover, at least for the SU(2) sector, the higher interactions must be constrained so we do not spoil the quadratic dispersion relation. For example, suppose we had a higher interaction term like $c_1 g_{\text{YM}}^4 \text{Tr}|[Z, [Z, Y]]|^2$. This would modify our dispersion relation as,

$$\omega_{ij} = \sqrt{1 + \frac{\lambda}{(2\pi)^2}|z_i - z_j|^2 + c_1 \lambda^2 |z_i - z_j|^4} \rightarrow \sqrt{1 + \frac{\lambda}{\pi^2}\sin^2\left(\frac{p}{2}\right) + 16 c_1 \lambda^2 \sin^4\left(\frac{p}{2}\right)} \ . \tag{5.18}$$

We believe that this restriction will come naturally from using the supersymmetry that underlines the derivation of the reduced matrix model. In fact, the magnon dispersion relation that enters the Bethe Ansatz was originally derived using SUSY alone [46]. It would be very interesting to integrate out higher spherical harmonics directly from SYM on $\mathbb{R} \times S^3$, and confirm that SUSY restricts the form of the resulting reduced matrix model (at least to leading order in the Yang-Mills coupling).

Now, note that keeping fixed the distance between the eigenvalues is just the Hofman-Maldacena limit [22] and it was the limit studied above. As an example, consider the quartic interaction $V_4 \sim g_{\text{YM}}^2 \text{Tr}|[Y_\alpha, Y_\beta]|^2$. It is easy to see that this will naturally be of $\mathcal{O}(1)$ under this limit, while the quadratic and cubic interactions are of $\mathcal{O}(\sqrt{\lambda})$. It would be interesting to compute the one-loop correction to the matrix model (3.9) for the SU(2) sector and verify that the new interaction is indeed suppressed.

On the other hand, we can try to define the more familiar BMN limit using the matrix model. In this case we need to take $\lambda|\vec{x}_i - \vec{x}_j|^2 = \text{fixed} \ll 1$. This is the limit where we consider short string bits first and *then* take the $\lambda \rightarrow \infty$ limit. For example, in the SU(2) sector we first take $L \rightarrow \infty$ first and so, $|\vec{x}_i - \vec{x}_j|^2 \rightarrow |\partial_\sigma \vec{x}|^2/L^2 \sim \mathcal{O}(1/L^2)$. We see that in this limit every higher commutator interaction to the matrix model will be relevant and thus our quadratic approximation is invalidated. This explains why our model is so much different from the usual one-loop spin chain. It is because our model is well defined in the *opposite* limit.

Nevertheless, the lack of interactions between the Y impurities in the SU(2) model makes it possible to match the string theory result at one loop in λ as we did in section 3. At higher loops we do not know if the matching requires corrections to the quadratic Hamiltonian.

What about $1/n$ corrections? One can hope to get a better understanding of these in the SU(2) sector. For that we need to take into account the backreaction δZ in the matrix model. However it can be that these corrections are entangled with the $1/\sqrt{\lambda}$ corrections. In any case, expanding around the normal matrix background should be regarded as an asymptotic expansion valid for BPS condensates with large angular momentum. It would be interesting to study this issue further.

6. Discussion

In this article we have attempted to clarify the relation between the reduced matrix model approach to calculate anomalous dimensions, and the usual operator mixing problem. This was done in terms of what we called "BPS condensates". These are long words in the scalar operators that look like sections of an otherwise BPS operator. In the matrix model these condensates were interpreted as a classical background which we use to define the perturbative expansion. In the dual string theory they represent (in a particular gauge) a infinitesimal section of a string with large angular momenta that localizes on a null geodesic on the S^5. Our method of expanding around a background of normal commuting matrices turns out to be a good approximation in the limit of large angular momentum on the sphere.

This interpretation allow us to match some well known string theory results in the limit where the condensates carry infinite angular momentum. For the SU(2) sector we were able to match the sigma model Hamiltonian of the dual string and its canonical structure with the matrix model Hamiltonian of the quadratic fluctuations around the 1/2 BPS condensates. This was done in the limit where we have giant magnons [22] and fast rotating strings [5]. The constrained canonical structure found in the matrix model also made clear why the infinite momentum limit correspond to a localization of the string on the equatorial S^1 of the S^5. Corrections to this limit are harder to understand since they require an understanding of the backreaction of the BPS condensates and perhaps the $1/\sqrt{\lambda}$ corrections.

For the SU(3) sector the matching was limited by the fact that we need to understand the backreaction to the BPS condensates in the matrix model even in the limit of infinite angular momentum. Nevertheless, we were able to match the canonical structure and the presence of SU(3) giant magnons that are in a a sense "rotations" of the usual SU(2) giant magnon. Finally, we explained why the reduced matrix model is more naturally defined in the strong coupling limit of the gauge theory.

So far, significant evidence has been accumulated that the effective Hamiltonian for states of $\mathcal{N} = 4$ on $\mathbb{R} \times S^3$ dual to holomorphic scalar operators on $\mathbb{R}^4$ is described by a reduced model of matrix quantum mechanics [12, 16, 17, 19–22]. What is really needed at this moment is a formal *derivation* of the matrix model at strong coupling. We believe that the secret lies in expanding around (nearly) highly supersymmetric states. This is where the BPS condensates can be very useful, specially in the infinite momentum limit. For example, for 1/2 BPS condensates of infinite angular momentum, we can focus on a single transverse excitation Y in an infinite "sea" of Z fields: $\mathcal{O} \sim \cdots ZZZZYZZZ \cdots$. The Feynman diagrammatics should greatly simplify by the fact that if the Y was changed for a Z field, all the diagrams must add to zero by supersymmetry. Of course, it would be nice to derive the matrix model without resorting to the usual diagrammatic calculations but instead integrating out higher spherical harmonics on the S^3.

Understanding the operator mixing problem in terms of a reduced matrix model can be used as an alternative route to using integrability in testing the AdS/CFT correspondence. This is because, as we saw, one can match directly the Hamiltonian of the dual string

and its canonical structure instead of having to find its spectrum. Nevertheless it would be interesting to understand the emergence of integrable structures in the language of the reduced matrix model. In fact, it is very useful to calculate the scattering phase for the string bits in the SU(2) sector with the quadratic Hamiltonian (3.28). Perhaps one could match the phase calculated in [22] using the sine-Gordon model in certain limit.

Acknowledgments

A. Anti-normal ordering

In this section we prove the identity,

$$\int \frac{d\Omega_3}{\text{Vol}(S^3)} \psi_{n_1,n_2}(Y,Z)^* Z^n \bar{Z}^m Y^k \bar{Y}^l \psi_{n_1',n_2'}(Y,Z) = \langle n_1, n_2 |_\circ^\circ (\hat{Z}^\dagger)^n \hat{Z}^m (\hat{Y}^\dagger)^k \hat{Y}^l {}_\circ^\circ |n_1', n_2' \rangle , \tag{A.1}$$

where the operators in the r.h.s. were defined in (4.10). Now, we can always identify the result of the integration with an effective operator,

$$[Z^{n_1} \bar{Z}^{m_1} Y^{n_2} \bar{Y}^{m_2}]_{\text{int}} |\tilde{n}_1, \tilde{n}_2 \rangle \simeq \frac{c_{\tilde{n}_1+n_1-m_1,\tilde{n}_2+n_2-m_2} c_{\tilde{n}_1,\tilde{n}_2}}{(c_{\tilde{n}_1+n_1,\tilde{n}_2+n_2})^2} |\tilde{n}_1+n_1-m_1, \tilde{n}_2+n_2-m_2 \rangle , \tag{A.2}$$

where,

$$c_{n,m} = \sqrt{\frac{(2+n+m)!}{2n!m!}} . \tag{A.3}$$

All we need to do now is to match the r.h.s. of (A.2) with the result of the anti-normal ordered form of the dual operators (4.10). For simplicity we will do this only in the case where $n_i - m_i > 0$. The other cases follow similarly. For the integrations we will use the identity,

$$c_{\tilde{n}_1+n_1,\tilde{n}_2+n_2} = c_{\tilde{n}_1,\tilde{n}_2} \prod_{k=1}^{n_1} \frac{1}{\sqrt{\tilde{n}_1+k}} \prod_{l=1}^{n_2} \frac{1}{\sqrt{\tilde{n}_2+l}} \prod_{s=1}^{n_1+n_2} \sqrt{2+\tilde{n}_1+\tilde{n}_2+s} . \tag{A.4}$$

Therefore, we can write the r.h.s. of (A.2) as,

$$\begin{aligned}
\frac{c_{\tilde{n}_1+n_1-m_1,\tilde{n}_2+n_2-m_2} c_{\tilde{n}_1,\tilde{n}_2}}{(c_{\tilde{n}_1+n_1,\tilde{n}_2+n_2})^2} = &\prod_{k=1}^{n_1-m_1} \sqrt{\tilde{n}_1+k} \prod_{k=n_1-m_1+1}^{n_1} (\tilde{n}_1+k) \\
&\times \prod_{l=1}^{n_2-m_2} \sqrt{\tilde{n}_2+l} \prod_{l=n_2-m_2+1}^{n_2} (\tilde{n}_2+l) \\
&\times \prod_{s=1}^{n_1-m_1+n_2-m_2} \frac{1}{\sqrt{2+\tilde{n}_1+\tilde{n}_2+s}} \\
&\times \prod_{s=n_1-m_1+n_2-m_2+1}^{n_1+n_2} \frac{1}{2+\tilde{n}_1+\tilde{n}_2+s} . \tag{A.5}
\end{aligned}$$

It is now straightforward to verify that one gets the same result using the anti-normal ordering form of the operators (4.10): $\hat{Z}^{m_1}\hat{Y}^{m_2}(\hat{Z}^\dagger)^{n_1}(\hat{Y}^\dagger)^{n_2}|\tilde{n}_1, \tilde{n}_2\rangle$ in the case $n_i - m_i > 0$.

B. SU(3) string action

Here we derive the action for fast rotating strings in the SU(3) sector (4.24). As usual, we start with the Polyakov action in momentum space [40],

$$
S_p = \frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \left(p_\mu \partial_0 x^\mu + \frac{1}{2} A^{-1} \left[G^{\mu\nu} p_\mu p_\nu + G_{\mu\nu} \partial_1 x^\mu \partial_1 x^\nu \right] + B A^{-1} p_\mu \partial_1 x^\mu \right),
$$
(B.1)

where $A = \sqrt{-g}g^{00}$, $B = \sqrt{-g}g^{01}$ and g^{ab} is the worldsheet metric.

We now consider string moving in $\mathbb{R} \times S^5$ with the following parametrization:

$$
t = \tau, \quad X = \sqrt{\frac{\eta}{1+\eta}} e^{i\varphi}, \quad Y = \frac{r_2}{\sqrt{1+\eta}} e^{i(\tau+\phi_2)}, \quad Z = \frac{r_1}{\sqrt{1+\eta}} e^{i(\tau+\phi_1)}.
$$
(B.2)

The metric in these coordinates read,

$$
ds^2 = -\frac{\eta}{1+\eta} dt^2 + \frac{1}{1+\eta} \left[2dt(r_1^2 d\phi_1 + r_2^2 d\phi_2) + \frac{d\eta^2}{4\eta} + \sum_{\alpha=1,2} (dr_\alpha^2 + r_\alpha^2 d\phi_\alpha^2) + \eta d\varphi^2 \right].
$$
(B.3)

We want to use the remaining gauge freedom to distribute the angular momentum in φ uniformly along the string. This is appropriate to compare with the operators (2.7). We have that,

$$
L = \frac{\sqrt{\lambda}}{4\pi} \int_0^{2\pi} d\sigma p_\varphi = \frac{\sqrt{\lambda}}{2} p_\varphi.
$$
(B.4)

We want to expand the action at first non-trivial order at large p_φ.

The Virasoro constraints that follow from (B.1) are,

$$
G^{\mu\nu} p_\mu p_\nu + G_{\mu\nu} x'^\mu x'^\nu = 0,
$$
(B.5)

$$
p_\mu x'^\mu = 0.
$$
(B.6)

We can now solve for p_t using (B.5),

$$
p_t = p_{\phi_1} + p_{\phi_2} - \sqrt{\Lambda + G_{ij} x'^i x'^j + \left(\frac{1+\eta}{\eta} \right) p_\varphi^2},
$$
(B.7)

where,

$$
\Lambda = (p_{\phi_1} + p_{\phi_2})^2 + G^{ij} p_i p_j + \left(\frac{\eta}{1+\eta} \right) \frac{p_i x'^i}{p_\varphi^2},
$$
(B.8)

and $i, j = \eta, r_1, r_2, \phi_1, \phi_2$ and we have used (B.6) to solve for φ'.

We can now plug the value of p_t calculated above back into the action. We get an effective action in terms of the momenta p_i. Since the momenta enter only algebraically

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into the action, we can easily solve for them using their equations of motion. Plugging the result back into the action we get,

$$S_p = -\frac{\sqrt{\lambda}}{4\pi} \int d\tau \int_0^{2\pi} d\sigma \sqrt{\left(G_{ij}x'^i x'^j + \frac{1+\eta}{\eta}p_\varphi^2\right)(1 - g_{ij}\dot{x}^i\dot{x}^j)} \,, \qquad (B.9)$$

where,

$$g_{ij} = \frac{1}{2}\frac{\partial^2\Lambda}{\partial p_i \partial p_j} \,, \qquad (B.10)$$

and $\dot{x}^{\phi_\alpha} \equiv 1 + \dot{\phi}_\alpha$.

One can make a systematic expansion at large p_φ where one gets an effective action which is linear in the $\dot{x}^i$ and one eliminates higher powers of the time derivatives in terms of higher spatial derivatives [40, 41]. Here we do not need to follow this procedure in detail since we want the leading order at large p_φ. Therefore, as usual we assume that all time derivatives are of order $\sim 1/p_\varphi^2$. Then expanding the action (B.9) at leading non-trivial order we find the result (4.24). Higher order corrections will only affect the form of the Hamiltonian but not the canonical structure.

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